Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
https://doi.org/10.47509/JSCS.2026.v05i01.04
https://www.arfjournals.com/jscs

Improved Control Charts with Two Levels of Inspection to Detect Changes in the Location Parameter

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Received:	Revised:	Accepted:	Published:
March 13, 2026	June 13, 2026	June 15, 2026	June 30, 2026

Abstract

For a univariate / multivariate production process, an ‘Improved Control Chart with Two Levels of inspection’ to detect changes in the target value (*ICCTL)* is proposed. For univariate processes, target value may be the processes mean, process fraction defective etc. For multivariate processes, target value may be a mean vector. Like in MIL-STD-105-E sampling plan, two levels of inspections namely ‘Reduced’ and ‘Tightened’ are used to develop *ICCTL*. Corresponding expressions for ‘Average Run Length’ (*ARL*) and ‘Average Time to Signal’ (*ATS*) of *ICCTL* are obtained. It is numerically illustrated that *ICCTL* performs significantly better as compared to the related control charts.

Key Words: Control Limits, Average Run Length, Average Time to Signal, Synthetic control chart, *GR* control chart

MSC 2020 subject classification: 62P30

1. Introduction

Shewhart type control charts are quite effective in detecting large shifts and are insensitive for small to moderate shifts in the target value. According to this type of control chart, a process is considered to be in control, if a charting statistic falls within the control limits. In the usual charts like $\overline{X}$ chart, χ^2 and T^2 charts, *np* and *C* charts; the decision about status of the process depends only on the most recent value of the charting statistic. In such type of control charts, the process is considered as in control, if the current value of the charting statistic is within the control limits irrespective of its departure from the target value. For such types of control charts, a sample point

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(value of charting statistic) being close to the control limits, though within the control limits, gives a stronger evidence to suspect the quality of the production process. Thus, it is desirable to consider the relative position of a point in assessing quality of the production process. This aspect is accounted in control charts with warning limits. By using the relative position of a point, Mahadik (2013 a) developed Shewhart $\bar{X}$ chart with varying warning and control limits. While developing his control chart, the idea of variable warning limits and variable control limits have been used. Li et al. (2014) proposed various methods of the computations of *ARL* and *ATS*. Some of them are Markov chain approach and integration equation method.

Towards improving the performance of the $\bar{X}$ chart, researchers have proposed efficient charts. To mention a few are 'Cumulative Sum' (*CUSUM*) control chart by Page (1954), 'Exponentially Weighted Moving Average' (*EWMA*) control chart, originally proposed by Roberts (1959). Lucas and Saccucci (1990) have compared the *EWMA* chart with the *CUSUM* chart and conclude that there is a little practical difference between the two techniques.

Bourke (1991) introduced a 'Conforming Run Length' (*CRL*) chart for increases in fraction non-conforming. Wu and Spedding (2000) introduced a Synthetic control chart, which is a combination of $\bar{X}$ chart and a *CRL* chart. This is one of the run-length based control charts. Gadre and Rattihalli (2004, 2006) introduced 'Group Runs' (*GR*) control chart and 'Modified Grup Runs' (*MGR*) control charts, which are the run length-based control charts.

For multivariate processes, Hotelling (1947) developed a T^2 and χ^2 charts. Mahadik (2013 b) proposed 'Variable Sampling Interval and Warning Limits to monitor the process mean Vector' (*VSIWL*-T^2) chart by using sampling interval as well the warning limits. Lowry et al. (1992) proposed a 'Multivariate *EWMA*' (*MEWMA*) control chart. Prabhu and Runger (1997) carried out a detailed analysis of the *ARL* performance of the *MEWMA* control chart. Crosier (1988) and Pingatiello and Runger (1990) have explained several 'Multivariate CUSUM' (*MCUSUM*) procedures.

Rattihalli et al. (2021-2022) introduced a 'Modified Control Chart with Warning Limits' (*MCCWL*) to monitor the target value of a production process. Gadre and Rattihalli (2022) introduced an 'Improved Control Chart to monitor the Location parameter' (*ICC-L*). Gadre and Patel (2024) introduced 'Improved Warning Limits control chart to detect shifts in the process mean' (*IWL*-$\bar{X}$) chart. Gadre (2025-2026) proposed 'Improved Warning Limits Control Chart for increases in fraction nonconforming' (*IWL-d*) chart. Also, Gadre (2025) introduced 'Improved Warning Limits control chart to detect changes in the mean vector' (*IWL*-χ^2) chart.

It is numerically illustrated that *ATS* performance of *IWL*-$\bar{X}$ chart is better as compared to *ICC*-$\bar{X}$ and *MCCWL*-$\bar{X}$. Very similar results are observed related to *IWL*-χ^2 and *IWL-d* charts.

Some acceptance sampling plans are used to sentence the lot. Some of them are single sampling plans, double sampling plans etc. In MIL-STD-105E acceptance sampling plan, three

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https://www.arfjournals.com/jscs

levels of inspections, namely, reduced, normal and tightened are used. Two levels of inspection; 'Reduced' and 'Tightened' are used to develop the proposed *ICCTL*. Let T be the charting statistic. Let the first and second level of inspection be 'Reduced' and 'Tightened' respectively. For r^{th} (r = 0, 1, 2...) sample of size n each, let T_r be the related charting statistic. The stopping rule of *ICCTL* is, Declare the process as out of control if, T_1 falls outside the control limits of second level of inspection or for $r > 1$, if T_r falls outside the control limits of the first level of inspection and $T_{(r+1)}$ falls outside the control limits of second level of inspection'. It is numerically observed that *ICCTL* performs much better compared to *MCCWL*, *ICC* and *ICCWL*.

Remainder of this article is organized as follows. **Section-2** covers brief review of the related control charts. **Section-3** covers the basic notations of *ICCTL*. Operation and the design to derive the *ATS* expression of *ICCTL* are also given in the same section. In **Section-4**, additional notations of the 'Improved Control chart with Two Levels of inspection to detect shifts in the process mean' ($ICCTL - \bar{X}$) are introduced. Also, some numerical illustrations and comparative study of $ICCTL - \bar{X}$ with $IWL - \bar{X}$ chart, $ICC - \bar{X}$, $MCCWL - \bar{X}$ and $\bar{X}$ chart is carried out in the same section. **Section-5** covers the additional notations, the *ATS* expression, numerical illustrations and comparative study of the *ICCTL-d* with the *IWL-d* chart, *ICC-d*, *MCCWL-d* and the *np* chart. For multivariate processes, the notations and the *ATS* expression of the *ICCTL*-χ^2 is given in **Section-6**. Numerical illustrations are considered and comparative study of *ICCTL*-χ^2 with *IWL*-χ^2 chart, *ICC*-χ^2, *MCCWL*-χ^2 and the χ^2 chart are carried out in the same section. Concluding remarks are given in **Section-7**.

2. Brief Review of the related charts

A brief review of *MCCWL*, *ICC-L* and *ICCWL* is given as below.

***2.1. Modified Control Chart with Warning Limits (MCCWL) to monitor the location parameter*:** Rattihalli et al. (2021-2022) developed this control chart. They used a chart with warning limits and based on it, a rule was proposed to stop the process. Let 'A', 'R' and 'W' be respectively the acceptance region, rejection region and the warning regions of this chart. The stopping rule for *MCCWL* is 'Declare the process as out of control if the sample point (T, say; value of the charting statistic) falls in R or two successive sample points fall in W'. Let T_r (r = 0, 1, …) be the r^{th} charting statistic. In *MCCWL*, at time zero, it is assumed that ($T_0 \in A$).

2.2 Improved Control Chart to monitor the Location parameter (ICC-L): Gadre and Rattihalli (2022) developed this control chart. Let T_r (r = 0, 1,) be the charting statistic as mentioned above. Also, 'A' and 'R' are the Acceptance and rejection regions respectively. Let 'S' be the sample space. Note that, (A U R) = S. Stopping rule for this chart is 'Stop if $T_1 \in R$; otherwise

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https://www.arfjournals.com/jscs

for $r > 1$, if $T_r \in R$ and $T_{(r+1)} \in R$'. If it is assumed that $T_0 \in R$, then this rule can be written as for r (r = 0, 1, ..), 'Declare the process as out of control if two successive charting statistics fall in $\underline{R}$.'

2.3. Improved Control Chart with Warning Limits' (ICCWL) to monitor the location parameter*:** Gadre and Patel (2024), Gadre (2025-26) and Gadre (2025) developed these control charts. Location parameters (L, say) are the process mean, the number of defectives in a sample of size n and the process mean vector respectively. Let 'A', 'W' and 'R' be the regions as mentioned in ***sub-section 2.1. It is assumed that, at time zero, $T_0 \in W$. Under this assumption, the stopping rule for *IWL-L* is 'Declare the process as out of control if $T \in R$ or two successive sample points $T \in W$'.

In the next section, the basic notations, operation and the design to derive the *ATS* expression of the proposed *ICCTL* is discussed.

3. Basic Notations, operation and the Design

Basic notations of the *ICCTL* are given below.

3.1. Basic Notations:

1. n_{itl}: Sample Size.
2. δ: Change in the location parameter.
3. δ_1: Change in the location parameter, the magnitude of which is considered large enough to seriously impair quality of the product.
4. M_0: In control location parameter.
5. M: The location parameter.
6. T_r: Charting statistic of the r^{th} (r = 0, 1, …) sample.
7. UCL_1: Upper control limit of the first level of inspection.
8. UCL_2: Upper control limit of the second level of inspection.
9. LCL_1 and LCL_2: Lower control limits of the respective levels of inspection.
10. $A = \{ T_r \in (LCL_1, UCL_1)\}$, $B = \{ T_{(r+1)} \in (LCL_2, UCL_2) \mid T_r \notin (LCL_1, UCL_1)\}$.
11. $A^c = \{ T_r \notin (LCL_1, UCL_1)\}$, $B^c = \{ T_{(r+1)} \notin (LCL_2, UCL_2) \mid T_r \notin (LCL_1, UCL_1)\}$.
12. $P(A)$, $P(A^c)$, $P(B)$ and $P(B^c)$ are respective probabilities
13. $ARL(\delta)$: $ARL_0 = ARL(0)$ and $ARL_1 = ARL(\delta_1)$: The related *ARLs*
14. $ATS(\delta)$: $n_{itl}(ARL(M))$. Similar meaning for ATS_0 and ATS_1,
15. τ: The minimum required value of ATS_0.

Next, the operation of *ICCTL* is discussed through the flow chart.

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3.2. The Operation:

It is evident that if the charting statistic $T \in A$, the process is under statistical control. There is no substantial evidence to suspect that the process might have gone out of control. To have such type of evidence, consider the status of the process based on the following rule.

Let T_r be the r^{th} (r = 0, 1,) charting statistic. It is assumed that $T_0 \notin A$. Then the above rule can be rewritten as, Declare the process as out of control if $T_1 \notin B$ or for some r (> 1), $T_r \notin A$ and $T_{(r+1)} \notin B$ for the first time. Figure 1 given below is useful to study the operation of *ICCTL*.

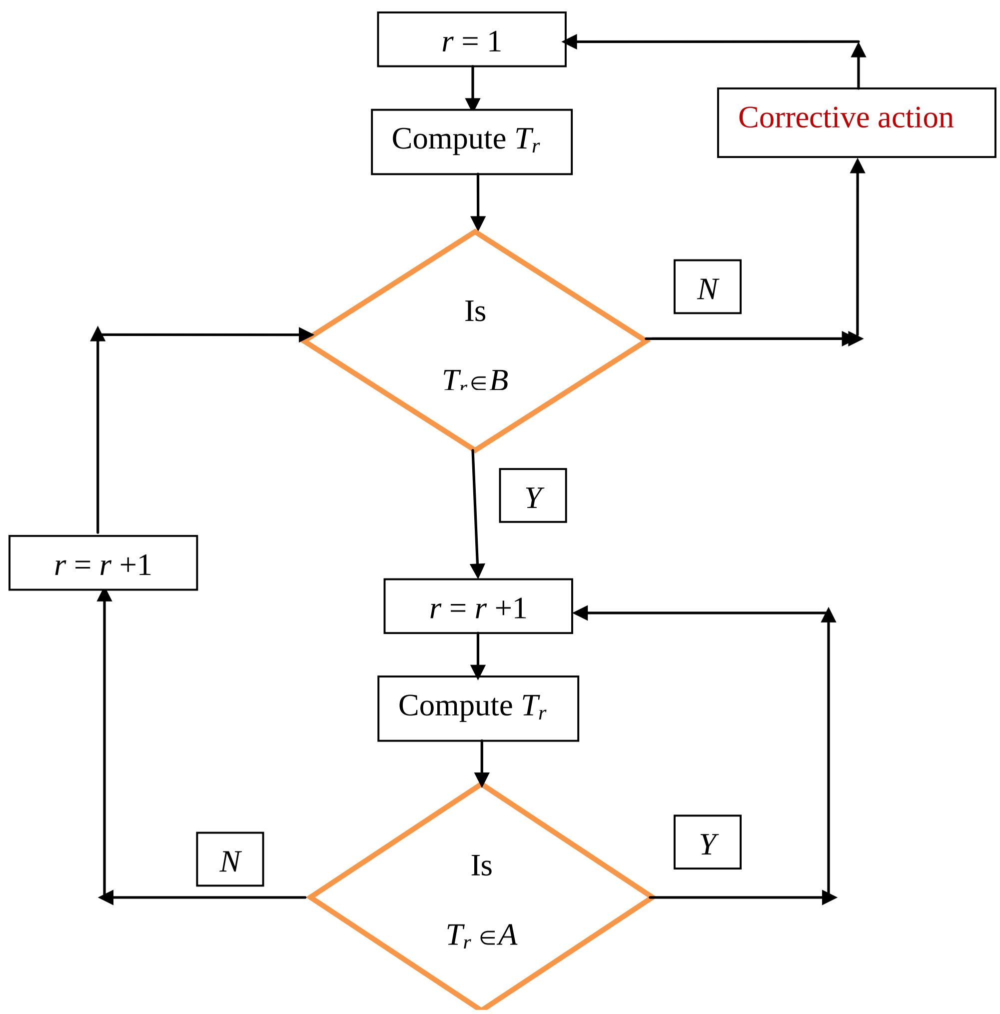

Figure 1: Flow-chart of *ICCTL*

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Now, design of the proposed chart is explained and *ATS* expression of the *ICCTL* is derived.

3.3. Design and the derivation:

As given in ***sub-section 3.1*** above, the non absorbing states are $\{A, A^c, B\}$ and B^c is an absorbing state and is 'Signal'. Now, the related t.p.m. $\boldsymbol{P}$ is

$$\boldsymbol{P} = \begin{array}{c c} & \begin{array}{cccc} A^c & A & B & Sig. \end{array} \\ \begin{array}{c} A^c \\ A \\ B \\ Sig \end{array} & \begin{pmatrix} 0 & 0 & P(B) & P(B^c) \\ P(A^c) & P(A) & 0 & 0 \\ P(A^c) & P(A) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \qquad (1)$$

Let $\boldsymbol{P_1}$ be the matrix obtained by deleting the last row and last column of $\boldsymbol{P}$. As mentioned in Brook and Evans (1972), E($\underline{N}$) = $\underline{ARL}$ = $(\boldsymbol{I}\text{-}\boldsymbol{P_1})^{-1}\underline{1}$. Now,

$$\boldsymbol{I}\text{-}\boldsymbol{P_1} = \begin{pmatrix} 1 & 0 & -P(B) \\ -P(A^c) & P(A^c) & 0 \\ -P(A^c) & -P(A) & 1 \end{pmatrix}$$

and $|\boldsymbol{I}\text{-}\boldsymbol{P_1}| = P(A^c)P(B^c)$. Also,

$$(\boldsymbol{I}-\boldsymbol{P_1})^{-1} = \frac{1}{P(A^c)P(B^c)}\begin{bmatrix} P(A^c) & P(A)P(B) & P(A^c)P(B) \\ P(A^c) & 1-P(A^c)P(B) & P(A^c)P(B) \\ P(A^c) & P(A) & P(A^c) \end{bmatrix}. \qquad (2)$$

Finally,

$$\underline{\boldsymbol{ARL}} = \frac{1}{P(A^c)P(B^c)}\begin{bmatrix} P(A^c)+P(B) \\ 1+P(A^c) \\ 1+P(A^c) \end{bmatrix}. \qquad (3)$$

As 'A^c' is the initial state of the tpm $\boldsymbol{P}$ given in Equation (*1*), *ARL* of the *ICCTL* is $\frac{P(A^c)+P(B)}{P(A^c)P(B^c)}$. Hence *ATS* of the proposed chart is,

$$ATS = n\,\frac{P(A^c)+P(B)}{P(A^c)P(B^c)}\,. \qquad (4)$$

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To obtain the design parameters of *ICCTL* and to compare the *ATS* performance of *ICCTL* with the related charts, *ATS* model is to be used (Refer Gadre and Rattihalli (2004, 2006). In the next section, for the quality characteristic $X \sim N(\mu, \sigma^2)$ distribution, additional notations and the numerical illustrations of the $ICCTL - \bar{X}$ are studied.

4. $\boldsymbol{ICCTL - \bar{X}}$

As the quality characteristic $X \sim N(\mu, \sigma^2)$ distribution and the observations are independent, $\bar{X} \sim N(\mu, \sigma^2/n)$. Following are some additional notations to be used in $ICCTL - \bar{X}$.

4.1. Additional notations:

1. $T = \bar{X}$: The Sample mean.
2. For r = 0, 1, 0, 0; $T_r = \bar{X}_r$: The r^{th} sample mean.
3. $n_{itl\bar{x}}$: Sample Size.
4. μ: Process mean.
5. μ_0: In-control process mean.
6. $k_{\bar{x}}$: coefficient of the control limits of first level of inspection.
7. $k_{1\bar{x}}$: coefficient of the control limits of second level of inspection.
8. $LCL_{1it\bar{x}} = \mu_0 - k_{\bar{x}} \frac{\sigma}{\sqrt{n_{itl\bar{x}}}}, UCL_{1it\bar{x}} = \mu_0 + k_{\bar{x}} \frac{\sigma}{\sqrt{n_{itl\bar{x}}}}$.
9. $LCL_{2it\bar{x}} = \mu_0 - k_{1\bar{x}} \frac{\sigma}{\sqrt{n_{itl\bar{x}}}}, UCL_{2it\bar{x}} = \mu_0 + k_{1\bar{x}} \frac{\sigma}{\sqrt{n_{itl\bar{x}}}}$.
10. $A = \left\{ \bar{X}_r \in (LCL_{1it\bar{x}}, UCL_{1it\bar{x}}) \right\}$, $A^c = \left\{ \bar{X}_r \notin (LCL_{1it\bar{x}}, UCL_{1it\bar{x}}) \right\}$.
11. $B = \left\{ \bar{X}_{(r+1)} \in (LCL_{2it\bar{x}}, UCL_{2it\bar{x}}) \mid \bar{X}_r \notin (LCL_{2it\bar{x}}, UCL_{2it\bar{x}}) \right\}$.
12. $B^c = \left\{ \bar{X}_{(r+1)} \notin (LCL_{2it\bar{x}}, UCL_{2it\bar{x}}) \mid \bar{X}_r \notin (LCL_{2it\bar{x}}, UCL_{2it\bar{x}}) \right\}$.
13. $P(A)$, $P(A^c)$, $P(B)$ and $P(B^c)$ are respective probabilities.
14. *ATSs* and τ have similar meaning as mentioned in Sub-Section 3.1.

4.2. Numerical Illustrations:

Example 1:

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Let μ_0 = 0, σ = 1, δ_1 = 0.2 and τ = 2000. For these input parameters, values of the design parameters of $ICCTL-\bar{X}$, $IWL-\bar{X}$ chart, $ICC-\bar{X}$, $MCCWL-\bar{X}$ and $\bar{X}$ chart along with respective ATS_1 values are as follows.

1. **$\bar{X}$ chart**: $n_{\bar{x}}$ = 111, $k_{\bar{x}}$ = 1.915, ATS_1 = 192.6385
2. **$MCCWL-\bar{X}$**: $n_{m\bar{x}}$ = 107, $k_{1m\bar{x}}$ = 1.575, $k_{m\bar{x}}$ = 1.965, ATS_1 = 190. 9142
3. **$ICC-\bar{X}$**: $n_{i\bar{x}}$ = 97, $k_{i\bar{x}}$ = 1.226, ATS_1 = 162. 676
4. **$IWL-\bar{X}$ chart**: $n_{iw\bar{x}}$ = 91, $k_{1iw\bar{x}}$ = 1.315, $k_{iw\bar{x}}$ = 2.505, ATS_1 = 151.631342
5. **$ICCTL-\bar{X}$**: $n_{it\bar{x}}$= 41, $k_{1it\bar{x}}$ = 0.004, $k_{it\bar{x}}$ = 3.988, ATS_1 = 58.0775

This example shows that, ATS_1 and the sample size of $ICCTL-\bar{X}$ are significantly less than ATS_1 and the sample sizes of the other four charts.

Example1 (Cont.): For further comparison of $ICCTL-\bar{X}$ with the four related charts, various values of $\delta \leq 0.3$, the *ATS (δ)* values are computed. These values are given in Table 1. Normalized *ATS* values (normalized w.r.t. the $\bar{X}$ chart) against δ values related to Table 1 are given in Figure 2.

Table 1: *ATS* and normalized *ATS* values of the five charts

δ Values	ATS					Normalized ATS				
	$\bar{X}$ Chart	$MCCWL-\bar{X}$	$ICC-\bar{X}$	$IWL-\bar{X}$ Chart	$ICCTL-\bar{X}$	$\bar{X}$ Chart	$MCCWL-\bar{X}$	$ICC-\bar{X}$	$IWL-\bar{X}$ Chart	$ICCTL-\bar{X}$
0	2000.3	2000.7	2005	2001.2	2011.2	1	1.00019997	1.00235	1.000449933	1.003399
0.01	1925.5	1953	1960.5	1955.7	1941.4	1	1.014282005	1.0182	1.015684238	1.003687
0.02	1821.8	1822.3	1848	1829.5	1754.7	1	1.000274454	1.0144	1.004226589	0.953068
0.03	1638.2	1638.4	1682.4	1648	1503.5	1	1.000122085	1.027	1.005982176	0.902393
0.04	1434.5	1434	1488.1	1441	1239.4	1	0.999651446	1.037	1.004531196	0.846427
0.05	1235.5	1233.9	1287.6	1233.8	996.4	1	0.998704978	1.042169	0.998624039	0.790854
0.06	1055.1	1052.5	1097.4	1042.3	789.3	1	0.997535779	1.040091	0.987868448	0.738413
0.07	898.8428	895.2351	926.9	874.5521	620.6	1	0.995986284	1.031215	0.972975586	0.689998
0.08	766.9137	762.6142	779.9	732.4474	487	1	0.994393763	1.016933	0.955058437	0.645835
0.09	657.1115	652.4132	656.4	614.6395	382.8	1	0.992850072	0.998917	0.935365611	0.605833
0.10	566.34	561.512	554.5	518.22	302.2	1	0.991475086	0.979094	0.915033372	0.570152
0.11	491.4603	486.7203	471.1	439.8501	235	1	0.990355274	0.958572	0.894986024	0.538192
0.12	429.6475	425.1538	403.3	376.3338	193	1	0.989540961	0.938676	0.875912929	0.510418
0.13	378.5014	374.3561	348.2	324.8639	156.8	1	0.989048125	0.919944	0.858289824	0.486392
0.14	336.0429	332.3002	303.5	283.0876	129.2	1	0.988862434	0.903158	0.842415061	0.46631
0.15	300.6659	297.3418	267.1	249.0864	108.1	1	0.988944207	0.888361	0.828449119	0.449669
0.16	271.0766	268.1588	237.3	221.3208	92.1	1	0.989236253	0.875398	0.816451143	0.436408
0.17	246.236	243.6921	213	198.566	79.9	1	0.989668854	0.865024	0.806405237	0.42642
0.18	225.3092	223.0935	192.9	179.8517	70.5	1	0.990165959	0.856157	0.798243924	0.41898
0.19	207.6243	205.6831	176.4	164.4097	63.5	1	0.99065042	0.849612	0.79186155	0.41421
0.20	192.6385	190.9142	162.7	151.6313	58.1	1	0.991049037	0.844587	0.787128741	0.411652
0.21	179.9112	178.3456	151.3	141.0322	54	1	0.991297929	0.84097	0.783898946	0.410758
0.22	169.0836	167.6199	141.9	132.2256	50.9	1	0.991343335	0.83923	0.782013158	0.411631
0.23	159.8616	158.4461	134	124.901	48.5	1	0.991145466	0.838225	0.781307081	0.413483
0.24	152.0027	150.5863	127.4	118.807	46.7	1	0.990681744	0.838143	0.781611116	0.41644

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0.25	145.3059	143.8448	121.9	113.7392	45.3	1	0.989944662	0.83892	0.782756929	0.412922
0.26	139.6036	138.0598	117.4	109.53	44.3	1	0.988941546	0.840953	0.784578621	0.423342
0.27	134.7547	133.0966	113.6	106.0407	43.5	1	0.98769542	0.843013	0.786916523	0.426701
0.28	130.6402	128.8425	110.4	103.1557	42.9	1	0.986239305	0.845069	0.789616825	0.430189
0.29	127.1583	125.2021	107.8	100.7785	42.4	1	0.984616026	0.847762	0.792543625	0.434105
0.3	124.2219	122.0946	105.7	98.8272	42.1	1	0.982875	0.850897	0.795569863	0.437121

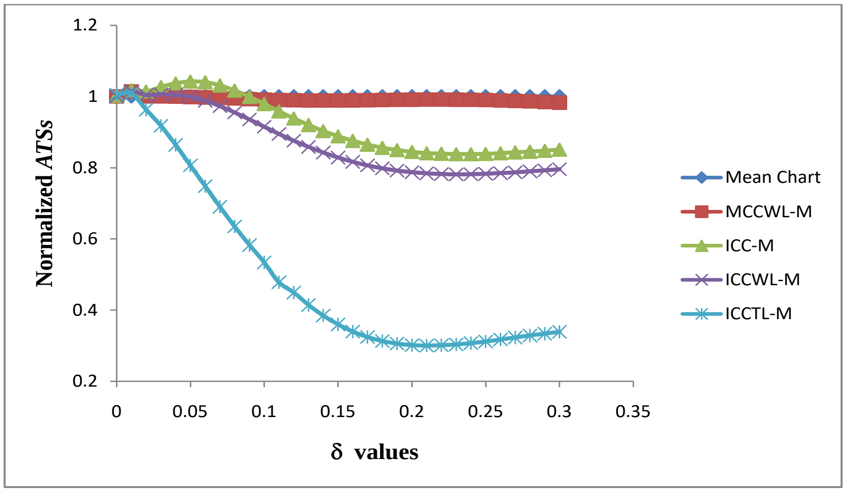

Figure 2: Graph of normalized *ATSs* against δ values of the five charts

From Table 1 and also from Figure2, it is much clear that, for $\delta > 0$, *ATS* (δ) of $ICCTL - \bar{X}$ is significantly less than the related four charts.

Example 2:

Here, an experiment similar to a 3^2-factorial experiment is carried out to compare the performance of the five control charts corresponding to 9 various combinations of input parameter values of (δ_1, τ). Following combinations of the input values are selected.

δ_1: 0.2, 0.5, 1
τ: 2000, 10000, 50000

Considering all possible 9 combinations of the input parameters (δ_1, τ) values, the design parameters along with respective ATS_1 values are computed for each of the five control charts and are given in Table2 below.

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Table 2: Design parameters of $\bar{X}$ chart, $MCCWL-\bar{X}$, $ICC-\bar{X}$, $ICCWL-\bar{X}$ and $ICCTL-\bar{X}$ with ATS_1 values

Ex.	δ_1	τ	$\bar{X}$ chart			$MCCWL-\bar{X}$				$ICC-\bar{X}$			$IWL-\bar{X}$ Chart				$ICCTL-\bar{X}$			
			$n_{\bar{x}}$	$k_{\bar{x}}$	ATS_1	$n_{m\bar{x}}$	$k_{1m\bar{x}}$	$k_{m\bar{x}}$	ATS_1	$n_{i\bar{x}}$	$k_{i\bar{x}}$	ATS_1	$n_{iw\bar{x}}$	$k_{1iw\bar{x}}$	$k_{iw\bar{x}}$	ATS_1	$n_{it\bar{x}}$	$k_{1it\bar{x}}$	$k_{it\bar{x}}$	ATS_1
1	0.2	2000	111	1.915	**192.6385**	107	1.575	1.965	**190.9142**	97	1.226	**162.676**	91	1.315	2.505	**151.6313**	41	0.004	3.988	**58.0775**
2	0.5	2000	32	2.41	**48.3250**	28	1.905	2.51	**47.2533**	24	1.602	**37.3445**	24	1.66	2.98	**35.2731**	11	0.016	3.972	**14.4800**
3	1	2000	11	2.78	**15.6198**	10	2.175	2.85	**15.1322**	8	1.858	**11.4993**	9	2.02	3	**11.5173**	4	0.04	3.992	**4.7664**
4	0.2	10000	188	2.35	**288.0927**	167	1.835	2.455	**282.2001**	148	1.548	**224.474**	141	1.615	2.955	**211.8496**	66	0.012	3.998	**85.3638**
5	0.5	10000	46	2.835	**64.7021**	39	2.165	2.95	**62.5997**	33	1.9	**47.3820**	37	2.135	3	**48.5108**	16	0.048	4	**19.7520**
6	1	10000	27	3.00	**27.3845**	27	2.9550	3	**19.0275**	10	2.15	**14.0284**	27	2.955	3	**27.3796**	5	0.142	4	**6.2715**
7	0.2	50000	272	2.78	**389.7184**	238	2.14	2.88	**377.7186**	196	1.862	**287.358**	213	2.045	3	**286.4017**	96	0.04	4	**118.3177**
8	0.5	50000	135	3.0	**135.3358**	135	2.955	3	**78.2709**	42	2.184	**57.5073**	135	2.955	3	**135.3358**	22	0.16	3.998	**25.9361**
9	1	50000	135	3.0	**135.3358**	135	2.955	3	**22.9871**	12	2.422	**16.5576**	135	2.955	3	**135**	7	0.402	4	**7.9884**

For all the 9 combinations we observe the following:

$n_{it\bar{x}}$ is the least among the related four charts.

ATS_1 of $ICCTL-\bar{X}$ significantly less as compared to the ATS_1 of four related charts.

Thus, conclude that, $ICCTL-\bar{X}$ performs much better as compared to the related charts.

Example2 (Cont.): A simulation study is carried out for all 9 combinations of the input parameters of Example 2. 10,00,000 sample means are generated for given input parameters and then by using the stopping rule of $ICCTL-\bar{X}$ simulated ATS_1 values are computed. These values are given in Table 3 below.

Table 3: Simulated and Analytical ATS_1 values of $ICCTL$-$\bar{X}$ chart for various δ_1 values

Ex.	δ_1	τ	Analytical ATS_1	Simulated ATS_1
1	0.2	2000	58.0775	59.8256
2	0.5	2000	14.4800	14.4792
3	1	2000	4.7664	4.7483

Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
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https://www.arfjournals.com/jscs

4	0.2	10000	85.3638	84.6446
5	0.5	10000	19.7520	19.7390
6	1	10000	6.2715	6.3071
7	0.2	50000	118.3177	119.3458
8	0.5	50000	25.9361	25.9218
9	1	50000	7.9884	8.0049

In Table 3, analytical ATS_1 means as given in Table 2. It means, using ATS model, those ATS_1 values are obtained for given input parameters.

From Table 3 it is observed that, for all 9 combinations of input parameters, simulated ATS_1 values are insignificantly different as compared to the corresponding analytical ATS_1.

Next section covers the development of the 'Improved Control Chart with Two Levels of inspection for increases in fraction nonconforming' (*ICCTL-d*).

5. ICCTL-d

This section covers the additional notations and for some numerical illustrations comparative study of *ICCTL-d* with the related four charts.

5.1. Additional notations of ICCTL-d: Following are additional notations of *ICCTL-d*.

1. n_{itd}: The sample size.
2. p_{itd}: The fraction nonconforming in a sample of size n_{itd}. Note that $d \sim b(n_{itd}, p_{itd})$.

For r = 0, 1, ..

3. d_r: The number of defectives in the r^{th} sample of size n_{itd}.
4. p_0: In control fraction nonconforming.
5. $ATS(p)$: Average number of units inspected by the proposed chart to detect changes in fraction nonconforming from in control value p_0 to p.
6. p_1: Design shift in the faction nonconforming, the magnitude of which is considered large enough to seriously impair the quality of the product.
7. $ATS_0 = ATS(p_0)$ and $ATS_1 = ATS(p_1)$.
8. c_{itd}: The acceptance number for the first level of inspection.
9. c_{1itd}: The acceptance number for the second level of inspection.
10. $UCL_{1itd} = c_{itd}$: Upper Control Limit of the first level of inspection.
11. $UCL_{2itd} = c_{1itd}$: Upper Control Limit of the second level of inspection.
12. $A = (d_r \leq c_{itd})$ and $B = (d_{(r+1)} \leq c_{1itd} \mid d_r > c_{itd})$.
13. $A^c = (d_r > c_{itd})$ and $B^c = (d_{(r+1)} > c_{1itd} \mid d_r > c_{itd})$.

Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
https://doi.org/10.47509/JSCS.2026.v05i01.04
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14. P(A), P(A^c), P(B) and *P(B^c)* are respective probabilities.

5.2. Numerical Illustrations:

Example 3:

For the input parameters $p_0 = 0.01$, $p_1 = 0.05$ and $\tau = 10000$, values of the design parameters of *ICCTL-d* and related four charts along with respective ATS_0 and ATS_1 values are given below.

1. ***d* chart**: $n_d = 77$, $c_d = 3$, $ATS_0 = 10152$, $ATS_1 = 142.3002$.
2. ***MCCWL-d***: $n_{md} = 73$, $c_{1md} = 2$, $c_{md} = 3$, $ATS_0 = 10120$, $ATS_1 = 136.0062$.
3. ***ICC-d***: $n_{iccd} = 41$, $c_{iccd} = 1$, $ATS_0 = 10187$, $ATS_1 = 108.5905$.
4. ***IWL-d* chart:** $n_{iccwld} = 38$, $c_{1iccwld} = 1$, $c_{iccwld} = 3$, $ATS_0 = 10536$, $ATS_1 = 100.1213$.
5. ***ICCTL-d***: $n_{icctld} = 41$, $c_{1icctld} = 0$, $c_{iccwld} = 2$, $ATS_0 = 10130$, $ATS_1 = 63.6130$.

This example clearly indicates that ATS_1 of *ICCTL-d* are significantly less as compared to the related four charts.

Example 3 (Cont.): Further, to study the behavior of *ICCTL-d* and the related four charts corresponding to the changes in *p* value, related *ATS* values and normalized *ATS* values are computed and are given in Table 4. Normalized *ATS* values (normalized w.r.t. *d* chart) against *p* values related to Table 4 are given in Figure 3.

Table 4: ***ATSs* and the Normalized *ATSs* of *d* chart, *MCCWL-d, ICC-d, IWL-d* chart and *ICCTL-d* for various values of *p***

P values	*ATS values*					*Normalized ATS values*				
	d chart	*MCCWL-d*	*ICC-d*	*IWL-d* chart	*ICCTL-d*	*d* chart	*MCCWL-d*	*ICC-d*	*IWL-d* chart	*ICCTL-d*
0.005	121760	133570	126320	134300	155670	1	1.096994	1.037451	1.102989	1.278499
0.006	62220	66880	64120	67777	75640	1	1.074896	1.030537	1.089312	1.215686
0.007	35580	37480	36430	38286	41140	1	1.053401	1.02389	1.076054	1.156268
0.008	22090	22830	22470	23487	24310	1	1.033499	1.017202	1.063241	1.100498
0.009	14610	14820	14760	15345	15310	1	1.014374	1.010267	1.050308	1.047912
0.01	10152	10120	10190	10536	10130	1	0.996848	1.003743	1.037825	0.997833
0.02	1119	1015	1050	1037	730	1	0.90706	0.938338	0.92672	0.652368
0.03	384	349	340	321.3803	190	1	0.908854	0.885417	0.836928	0.494792
0.04	208	194	170	158.4385	90	1	0.932692	0.817308	0.761724	0.432692
0.05	142	136	110	100.1213	60	1	0.957746	0.774648	0.70508	0.422535
0.06	112	109	80	73.5675	50	1	0.973214	0.714286	0.656853	0.446429
0.07	97	94	70	59.6213	50	1	0.969072	0.721649	0.614653	0.515464
0.08	88	86	60	51.617	40	1	0.977273	0.681818	0.586557	0.454545
0.09	83	80	50	46.7583	40	1	0.963855	0.60241	0.563353	0.481928
0.1	81	77	50	43.7038	40	1	0.950617	0.617284	0.539553	0.493827
0.11	79	76	50	41.7411	40	1	0.962025	0.632911	0.528368	0.506329
0.12	78	74	40	40.4635	40	1	0.948718	0.512821	0.518763	0.512821
0.13	78	74	40	39.6253	40	1	0.948718	0.512821	0.508017	0.512821

Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
https://doi.org/10.47509/JSCS.2026.v05i01.04
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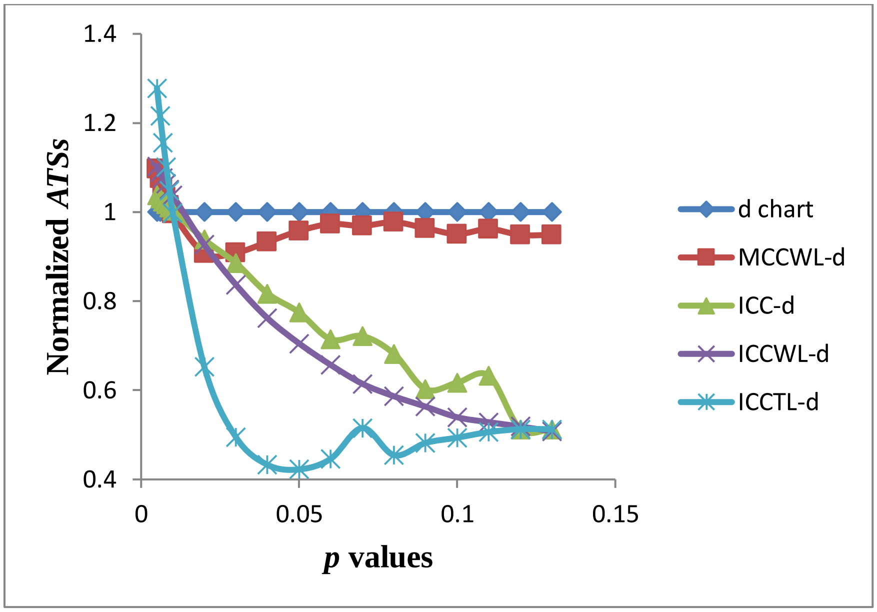

Figure 3: Graph of normalized *ATSs* against *p* values of the five charts

From Table 4 and also from Figure 3, it is much clear that, for $p \in [0, 0.12]$, *ATS*(p) of *ICCTL-d* is significantly less than the related four charts. For $p = 0.13$, *ATS*(p) of *ICCTL-d* and *ATS*(p) of *IWL-d* chart are almost same.

For $p < p_0$, *ATS*(p) of *ICCTL-d* is significantly larger than the related four charts. It means the proposed chart is efficient as compared to the related four charts.

Example 4:

In *ICCL-d* there are three input parameters namely (p_0, p_1, τ). Here two values of the input parameter p_0 and three values of the input parameters p_1 and τ are considered. Hence there are 18 combinations of the input parameters. Following combinations of the input parameters are selected.

p_0: 0.01, 0.005

p_1: $5p_0$, $20p_0$, $50p_0$,

τ : $10/p_0$, $50/p_0$, $100/p_0$.

To study the effect on the *ATS* performance of *ICCTL-d* with the related four charts is carried out. Considering all possible 18 combinations of the input parameters (p_0, p_1, τ), the values

Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
https://doi.org/10.47509/JSCS.2026.v05i01.04
https://www.arfjournals.com/jscs

of the design parameters, along with respective ATS_1 values are computed for each of the five control charts and are given in Table 5 below.

Table 5: Design parameters of the *d* chart, *MCCWL-d, ICC-d, IWL-d* chart and *ICCTL-d* with respective ATS_1 values

(p_0, p_1, τ)	*d* chart			*MCCWL-d*				*ICC-d*			*ICCWL-d*				*ICCTL-d*			
	n	c	ATS_1	n	c_1`	c	ATS_1	n	c	ATS_1	n	c_1`	c	ATS_1	n	c_1`	c	ATS_1
(0.01, 0.05, 1000)	24	1	70.7583	8	0	1	67.6615	11	0	59.1610	7	0	1	57.3804	36	0	1	55.1581
(0.01, 0.05, 5000)	41	2	121.6084	35	1	2	111.7982	54	1	93.6857	51	1	3	84.8027	46	0	2	62.6195
(0.01, 0.05, 10000)	77	3	142.3002	73	2	3	136.0062	41	1	108.5905	38	1	3	100.1213	41	0	2	63.6130
(0.01, 0.2, 1000)	9	1	15.9633	5	0	1	13.0979	6	0	11.0207	5	0	1	9.2919	6	0	0	12.8598
(0.01, 0.2, 5000)	5	1	19.0317	15	1	2	23.2454	2	0	15.4321	13	1	2	18.4729	8	0	1	12.8598
(0.01, 0.2, 10000)	17	2	24.6242	15	1	2	23.2454	15	1	21.6238	13	1	2	18.4729	8	0	1	12.8598
(0.01, 0.5, 1000)	4	1	5.8182	2	0	1	4.8000	2	0	3.5556	2	0	1	3.2000	2	0	2	4.2857
(0.01, 0.5, 5000)	4	1	5.8182	6	1	2	8.5622	2	0	3.5556	5	1	2	6.6321	2	0	0	4.2857
(0.01, 0.5, 10000)	3	1	6	6	1	2	8.5622	6	1	7.5642	5	1	2	6.6321	3	0	1	4.2857
(0.005, 0.025, 2000)	47	1	142.8090	16	0	1	135.3059	22	0	120.6214	14	0	1	114.9820	72	0	1	111.5573
(0.005, 0.025, 10000)	81	2	245.3342	69	1	2	225.9243	128	1	184.7282	101	1	3	171.3519	92	0	2	126.4313
(0.005, 0.025, 20000)	153	3	286.6547	145	2	3	273.9367	82	1	219.7465	76	1	3	201.4712	81	0	2	129.1769
(0.005, 0.1, 2000)	18	1	32.7442	10	0	1	26.8757	12	0	23.3052	10	0	1	19.3710	12	0	2	23.3052
(0.005, 0.1, 10000)	9	1	39.9717	3	0	1	39.7323	4	0	33.8217	3	0	1	31.9649	18	0	1	26.9615
(0.005, 0.1, 20000)	34	2	50.4087	29	1	2	47.5307	31	1	44.9521	26	1	2	38.0341	18	0	1	26.9615
(0.005, 0.25, 2000)	7	1	12.6114	4	0	1	10.3388	4	0	8.5598	4	0	1	7.2713	4	0	0	8.5598
(0.005, 0.25, 10000)	7	1	12.6114	3	0	1	10.6602	4	0	8.5598	3	0	1	7.4973	4	0	0	8.5598
(0.005, 0.25, 20000)	5	1	13.6170	12	1	2	18.3735	2	0	10.4490	10	1	2	14.5504	7	0	1	10.0211

From Table 5, the following results are observed.

For 13 of the 18 combinations considered, it is observed that ATS_1 values of the *ICCTL-np* are smaller compared to those of the remaining four charts.

For remaining 5 combinations of the 18, it is observed that, for given τ, to satisfy ATS_0 constraint in ATS model, ATS_0 values are much large as compared to the related τ values. See some related combinations. For 4th, 5th and 6th combinations, ATS_0 value is 35624; for 6th, 7th and 8th combinations, ATS_0 value is 32898; for 13thcombination, ATS_0 value is 5277.4 and for 16th, 17th and 18th combinations, ATS_0 value is 37985.

For multivariate processes, some Shewhart type multivariate control charts are χ^2 chart and T^2 chart proposed by Hotelling (1947), ' Modified Control Chart with Warning Limits to monitor mean vector' (*MCCWL*-χ^2) by Rattihalli et al. (2021-2022), 'Improved Control Chart to monitor the mean vector' (*ICC*-χ^2) by Gadre and Rattihalli (2022) and 'Improved Control Chart with Warning Limits to monitor mean vector' (*IWL*-χ^2 chart) by Gadre (2025). In the next section, 'Improved Control Chart with Two Levels of inspection to monitor the mean vector' (*ICCTL*-χ^2) is developed.

6. *ICCTL*-χ^2

Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
https://doi.org/10.47509/JSCS.2026.v05i01.04
https://www.arfjournals.com/jscs

This section covers the additional notations and for some numerical illustrations comparative study of *ICCTL-*χ^2 with the related four charts. Let p be the number of quality characteristics. For simplicity take $p = 2$ and let X, Y be the two quality characteristics.

6.1. Additional notations of ICCTL-χ^2: Following are additional notations of *ICCTL-*χ^2. Following are the notations related to *ICC-*χ^2.

1. $n_{it\chi 2}$: The sample size.
2. $\underline{M}_0$: $\underline{\mu}_0 = (\mu_{0x}, \mu_{0y})$: In control mean vector.
3. $\underline{M}_1$: $\underline{\mu}_1 = (\mu_{1x}, \mu_{1y})$: Out of control mean vector.
4. (σ_x, σ_y): The process variability of the respective quality characteristics.
5. Σ_0: The known covariance matrix, which depends on the correlation coefficient ρ.
6. $\underline{\delta}_1 = (\delta_{1x}, \delta_{1y}) = \left(\frac{\mu_{1x} - \mu_{0x}}{\sigma_{\bar{y}}}, \frac{\mu_{1y} - \mu_{0y}}{\sigma_{\bar{y}}}\right)$, where $\sigma_{\bar{X}} = \frac{\sigma_x}{\sqrt{n}}$, $\sigma_{\bar{Y}} = \frac{\sigma_y}{\sqrt{n}}$ for a bivariate process and it is shift in the standardized mean vector. Here, n is $n_{it\chi 2}$. For r = 0, 1, …
7. $\overline{\underline{X}}_r$: Sample mean vector of the r^{th} sample mean.
8. χ_r^2 : The test statistic of *ICCTL-*χ^2 for r^{th} sample.
9. $\chi_0^2 = n_{it\chi^2} (\overline{\underline{X}}_r - \underline{\mu}_0)' \Sigma_0^{-1} (\overline{\underline{X}}_r - \underline{\mu}_0)$: The test statistic for in control *ICCTL-*χ^2.
10. ρ_{xy}: The related correlation coefficient.
11. $k_{it\chi 2}$: Coefficient of the control limits of first level of inspection.
12. $k_{1it\chi 2}$: Coefficient of the control limits of second level of inspection.
13. $k_{it\chi 2}$ and $k_{1it\chi 2}$ are the upper control limits of the first and second level of inspection respectively.
14. $A = (\chi_r^2 \le k_{it\chi 2})$, $B = (\chi_{(r+1)}^2 \le k_{1it\chi 2} \mid \chi_r^2 > k_{it\chi 2})$,
15. $P(A)$, $P(A^c)$, $P(B)$ and $P(B^c)$ are respective probabilities.

Now, for some numerical illustrations considered, *ATS* performance of the *ICCTL-*χ^2 with the related four charts is carried out.

***6.2. Numerical Illustrations*:**

Example 5:

For $\underline{\delta}_1 = (0, 0.75)$, $\rho = 0.7$ and $\tau = 370$. For these input parameters, the design parameters and ATS_1 values of the proposed *ICCTL-*χ^2 chart and the related four charts are given below.

Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
https://doi.org/10.47509/JSCS.2026.v05i01.04
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(i) **χ^2 chart**: $n_{\chi 2} = 8$, $k_{\chi 2} = 7.67$, $ATS_1 = 12.3516$.
(ii) MCCWL-χ^2: $n_{mw\chi 2} = 8$, $k_{1m\chi 2} = 5.01$, $k_{m\chi 2} = 8.06$, $ATS_1 = 12.2129$.
(iii) ICC-χ^2: $n_{icc\chi 2} = 7$, $k_{i\chi 2} = 3.97$, $ATS_1 = 9.8603$.
(iv) IWL-χ^2 **chart**: $n_{iwch} = 7$, $k_{1iw\chi 2} = 4.01$, $k_{iw\chi 2} = 14.01$, $ATS_1 = 9.4702$.
(v) ICCTL-χ^2: $n_{it\chi 2} = 2$, $k_{1it\chi 2} = 0.01$, $k_{it\chi 2} = 21.03$, $ATS_1 = 1.7954$.

This example shows that ATS_1 and the sample values of the *ICCTL-*χ^2 is less than ATS_1 of the related four charts, which is an indicative of reduction of the delay in detection of the shift.

Example 5 (Cont.): Further, to study the behavior of *ICCTL-*χ^2 and the related four charts corresponding to the changes in δy value by fixing δx as zero, related *ATS* and normalized *ATS* values are computed and are given in Table 6. Normalized *ATS* values (normalized w.r.t. the χ^2 chart) against (δx, δy) values related to Table 6 are given in Figure 4.

Table 6: ***ATSs* and the Normalized *ATSs* of χ^2 chart, *MCCWL-*χ^2, *IWL-*χ^2 chart and *ICCTL-*χ^2 *when* $\delta_x = 0$ for various values of δ_y**

δ_x	δ_y	*ATS*					*Normalized ATS*				
		χ^2 chart	*MCCWL-*χ^2	*ICC-*χ^2	*IWL-*χ^2 chart	*ICCTL-*χ^2	χ^2 chart	*MCCWL-*χ^2	*ICC-*χ^2	*IWL-*χ^2 chart	*ICCTL-*χ^2
0	0	370.3474	370.1572	370.8917	370.0073	371.5766	1	0.999486	1.00147	0.999082	1.003319
0	0.1	282.3875	278.2856	287.3236	284.1366	300.2839	1	0.985474	1.01748	1.006194	1.063375
0	0.2	156.3721	149.4268	155.9337	151.8872	175.4179	1	0.955585	0.997196	0.971319	1.121798
0	0.3	82.4881	76.7462	76.5390	83.6985	87.0382	1	0.930391	0.927879	1.014674	1.055161
0	0.4	46.0817	42.4904	39.5630	37.8141	40.4683	1	0.922067	0.85854	0.820588	0.878186
0	0.5	28.1274	26.2113	22.8566	21.7643	18.6764	1	0.931878	0.81261	0.773776	0.663993
0	0.6	18.8930	18.0182	14.9696	14.2574	9.0433	1	0.953697	0.792336	0.754639	0.478659
0	0.7	13.9272	13.6241	11.0395	10.5624	4.9218	1	0.978237	0.792658	0.758401	0.353395
0	0.8	11.1724	11.1405	9.0087	8.6922	3.1959	1	0.997145	0.806335	0.778007	0.286053
0	0.9	9.6311	9.6943	7.9538	7.7550	2.4844	1	1.006562	0.825845	0.805204	0.257956
0	1	8.7865	8.8568	7.4219	7.3090	2.1945	1	1.008001	0.844694	0.831844	0.249758
0	1.1	8.3475	8.3950	7.1704	7.1144	2.0776	1	1.00569	0.858988	0.852279	0.248889
0	1.2	8.1379	8.1625	7.0619	7.0382	2.0308	1	1.003023	0.867779	0.864867	0.249548
0	1.3	8.0484	8.0588	7.0200	7.0115	2.0122	1	1.001292	0.872223	0.871167	0.250012
0	1.4	8.0149	8.0186	7.0057	7.0031	2.0048	1	1.000462	0.874085	0.87376	0.250134
0	1.5	8.0040	8.0051	7.0014	7.0008	2.0019	1	1.000137	0.874738	0.874663	0.250112

Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
https://doi.org/10.47509/JSCS.2026.v05i01.04
https://www.arfjournals.com/jscs

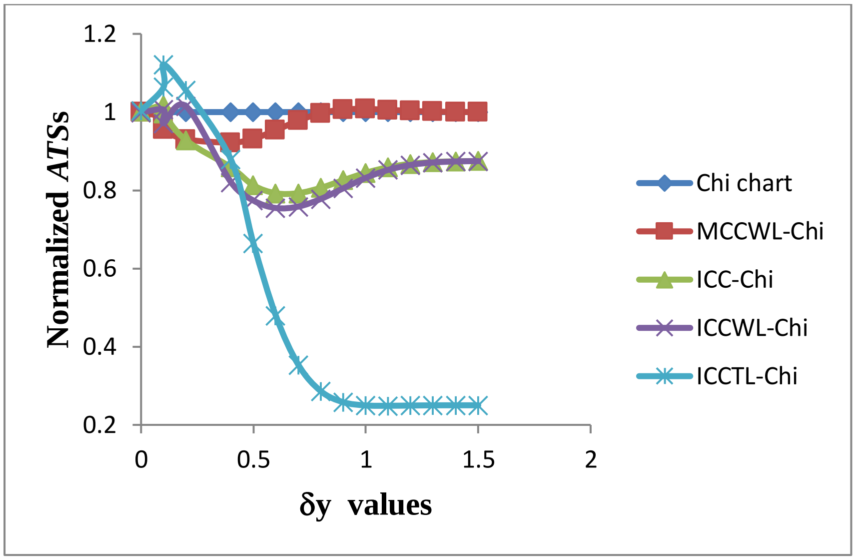

Figure 4: Graph of normalized *ATSs* against $\underline{\delta}$ values of the five charts

From Table 6 and Figure 4 it is observed that, for $\delta_y > 0.4$, normalized *ATSs* of *ICCTL-*χ^2 significantly less as compared to the related four charts. This is an indicative of *ICCTL-*χ^2 performs much better.

Example 6: Here, for $\rho = 0.7$, $\tau = 370$ and 14 combinations of $\underline{\delta}_1$, design parameters of the *ICCTL-*χ^2 and the related four charts are computed along with ATS_1 values of each of the combinations and are given in Table 7.

Table 7: Design parameters of χ^2 chart, *MCCWL-*χ^2, *ICC-*χ^2, *IWL-*χ^2 chart and *ICCTL-*χ^2 with ATS_1 values for $\rho = 0.7$

$\underline{\delta}_1 = (\delta_{1x}, \delta_{1y})$	χ^2 chart			*MCCWL-*χ^2				*ICC-*χ^2			*IWL-*χ^2 chart				*ICCTL-*χ^2			
	$\rho = 0.7$			$\rho = 0.7$				$\rho = 0.7$			$\rho = 0.7$				$\rho = 0.7$			
	n	*k*	ATS_1	*n*	k_1	*k*	ATS_1	*n*	*k*	ATS_1	*n*	k_1	*k*	ATS_1	*n*	k_1	*k*	ATS_1
(0, 0.50)	15	6.42	**23.2314**	14	5.01	6.66	**23.0071**	**12**	**3.43**	**19.0532**	11	4.01	8.63	**18.1887**	5	0.01	19.18	**6.7154**
(0, 0.75)	8	7.67	**12.3516**	8	5.01	8.06	**12.2129**	**7**	**3.97**	**9.8603**	7	4.01	14.01	**9.4702**	2	0.01	21.03	**3.8720**
(0, 1)	5	6.61	**7.7828**	5	6.01	8.82	**7.6284**	**4**	**4.53**	**6.0764**	4	5.01	10.81	**5.8039**	2	0.01	21.03	**2.1945**
(0, 1.50)	3	9.63	**3.9810**	3	7.01	9.76	**3.9395**	**2**	**5.23**	**3.0438**	2	6.01	11.55	**2.9716**	1	0.01	22.42	**1.0842**
(0.50, 0.50)	21	5.74	**33.9704**	20	4.01	6.12	**33.7879**	**18**	**3.03**	**28.5184**	19	3.01	10.2	**27.2821**	8	0.01	18.22	**10.2616**
(0.50, 0.75)	13	6.70	**20.8782**	13	5.01	6.84	**20.6684**	**11**	**3.52**	**17.0359**	10	4.01	9.16	**16.1572**	5	0.01	19.18	**6.0475**
(0.50, 1)	8	7.67	**12.5848**	8	5.01	8.06	**12.4229**	**7**	**3.97**	**10.0321**	7	4.01	14.01	**9.6285**	3	0.01	20.21	**3.5169**
(0.50, 1.50)	4	9.06	**5.7212**	4	6.01	9.37	**5.6416**	**3**	**4.82**	**4.4269**	3	5.01	12.92	**4.1939**	1	0.01	22.4200	**1.6023**
(0.75, 0.75)	12	6.86	**18.4397**	11	5.01	7.24	**18.2421**	**10**	**3.62**	**14.9806**	9	4.01	9.89	**14.1240**	4	0.01	19.63	**5.2229**
(0.75, 1)	9	7.44	**13.3028**	8	5.01	8.06	**13.1322**	**7**	**3.07**	**10.6222**	7	4.01	14.01	**10.1745**	3	0.01	20.21	**3.6843**
(0.75, 1.50)	5	8.61	**6.5748**	4	6.01	9.37	**6.4010**	**4**	**4.53**	**5.1406**	3	5.01	12.92	**4.8521**	1	0.01	22.42	**2.1002**
(1, 1)	8	7.67	**11.7290**	7	4.01	8.44	**11.6037**	**6**	**4.13**	**9.3409**	7	4.01	14.01	**9.0544**	2	0.01	21.03	**3.5022**
(1, 1.50)	5	8.61	**6.9357**	4	6.01	9.37	**6.8288**	**4**	**4.53**	**5.4161**	4	5.01	10.81	**5.2073**	1	0.01	22.42	**2.4292**
(1.50, 1.50)	4	9.06	**6.0804**	4	6.01	9.37	**5.9581**	**3**	**4.82**	**4.7287**	3	5.01	12.92	**4.4635**	1	0.01	22.42	**1.7954**

From Table 7, observed the following, For $\underline{\delta_1} \neq \underline{0}$,

Sample sizes of *ICCTL*-χ^2 are the least among those of the related four control charts.

For each of the 14 combinations of the input parameters, ATS_1 of *ICCTL*-χ^2 is significantly less as compared to the related four charts.

Also, for $\rho = 0.5$, $\tau = 370$, the design parameters and the input parameters along with the ATS_1 values are given in Table 8.

Table 8: Design parameters of χ^2 chart, *MCCWL*-χ^2, *ICC*-χ^2, *IWL*-χ^2 chart and *ICCTL*-χ^2 with ATS_1 values for $\rho = 0.5$

$\underline{\delta_1} = (\delta_{1x}, \delta_{1y})$	χ^2 chart			*MCCWL*-χ^2				*ICC*-χ^2			*IWL*-χ^2 chart				*ICCTL*-χ^2			
	$\rho = 0.5$			$\rho = 0.5$				$\rho = 0.5$			$\rho = 0.5$				$\rho = 0.5$			
	n	*k*	ATS_1	*n*	k_1	*K*	ATS_1	*n*	*k*	ATS_1	*n*	k_1	*k*	ATS_1	*n*	k_1	*k*	ATS_1
(0, 0.50)	19	5.94	**30.9906**	19	4.01	6.26	**30.8542**	**16**	**3.15**	**25.8896**	19	3.01	11.87	**25.1914**	7	0.01	18.49	**9.2309**
(0, 0.75)	11	7.04	**16.7419**	10	5.01	7.48	**16.5425**	**9**	**3.72**	**13.5130**	8	4.01	11.03	**12.7745**	3	0.01	20.21	**4.9639**
(0, 1)	7	7.94	**10.6136**	7	6.01	8.05	**10.4822**	**6**	**4.13**	**8.4243**	5	5.01	9.8	**8.2084**	2	0.01	21.03	**2.9582**
(0, 1.50)	4	9.06	**5.4827**	3	6.01	10.13	**5.4271**	**3**	**4.82**	**4.2280**	3	5.01	12.92	**4.0174**	1	0.01	22.42	**1.4852**
(0.50, 0.50)	19	5.94	**30.9906**	19	4.01	6.26	**30.8542**	**16**	**3.15**	**25.8896**	19	3.01	11.79	**25.1914**	7	0.01	18.49	**9.2309**
(0.50, 0.75)	13	6.70	**20.3251**	12	5.01	7.03	**20.1340**	**11**	**3.52**	**16.5768**	10	4.01	9.16	**15.6954**	4	0.01	19.63	**5.8924**
(0.50, 1)	9	7.44	**13.3533**	8	5.01	8.06	**13.1842**	**7**	**3.97**	**10.6660**	7	4.01	14.01	**10.2151**	3	0.01	20.21	**3.6975**
(0.50, 1.50)	5	8.61	**6.7608**	4	6.01	9.37	**6.6227**	**4**	**4.53**	**5.2821**	3	5.01	12.92	**5.0507**	1	0.01	22.42	**2.2667**
(0.75, 0.75)	11	7.04	**16.7419**	10	5.01	7.48	**16.5425**	**9**	**3.72**	**13.5130**	8	4.01	11.03	**12.7745**	3	0.01	20.21	**4.9639**
(0.75, 1)	8	7.67	**12.5371**	8	5.01	8.06	**12.3800**	**7**	**3.97**	**9.9969**	7	4.01	14.01	**9.5959**	2	0.01	21.03	**3.9874**
(0.75, 1.50)	5	8.61	**6.9472**	4	6.01	9.37	**6.84236**	**4**	**4.53**	**5.4250**	4	5.01	10.81	**5.2153**	1	0.01	22.44	**2.4401**
(1, 1)	7	7.94	**10.6136**	7	6.01	8.05	**10.4822**	**6**	**4.13**	**8.4243**	7	5.01	9.80	**8.2084**	2	0.01	21.03	**2.9582**
(1, 1.50)	5	8.61	**6.7608**	4	6.01	9.37	**6.6227**	**4**	**4.53**	**5.2821**	3	5.01	12.92	**5.0507**	1	0.01	22.42	**2.2667**
(1.50, 1.50)	4	9.06	**5.4827**	3	6.01	10.13	**5.4271**	**3**	**4.82**	**4.2280**	3	5.01	12.92	**4.0174**	1	0.01	22.42	**1.4852**

From Table 8, these three results indicate that *ICCTL*-χ^2 is much more efficient as compared to the four related charts.

7. Conclusions

The proposed $ICCTL - \bar{X}$ performs significantly better as compared to the $\bar{X}$ chart, $MCCWL - \bar{X}$, $ICC - \bar{X}$ and $ICCWL - \bar{X}$. For all combinations of the input parameters considered, ATS_1 of $ICCTL - \bar{X}$ is significantly less than that of the related four charts. Moreover, for positive δ shifts, ATS (δ) of $ICCTL - \bar{X}$ is much smaller than those of the related four charts.

For the multivariate processes, sample sizes as well ATS_1 values of *ICCTL*-χ^2 are the least among those of the related four control charts.

Some of the combinations of the input parameters *ICCTL-d*, ATS_1 values of the *ICCTL-d* are much smaller as compared to those of the related four charts.

Implementation/operation of the stopping rule of the process for the proposed charts is not much difficult. As these charts give scope to account for doubtful situations, and decisions are taken by getting additional information, these charts seem to be used by the industries.

Acknowledgement

I am thankful to the Referee and the Editor in Chief for valuable comments, which helped to improve the manuscript significantly. Moreover, I am thankful to the Editor in Chief for allowing me significant extra time to submit the revised manuscript.

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Journal of Statistics and Computer Science
Vol. 5 No: 1 (2026); pp 75-94 -ISSN: 2583-5068 - (c) ARF India
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